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Did Minkowski Change his Mind about Noneuclidean Symmetry in Special Relativity?

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Minkowski observed in a 1907 lecture that the symmetry of relativistic velocity space is the same as that of noneuclidean geometry. He withheld this text from publication, but Sommerfeld had it published 6 years after he died [1]. Six weeks later, Minkowski submitted a long, careful article [2], where he made only a much weaker statement about the noneuclidean parallel. In [3] he avoided the issue entirely. His reasons for the changes were long unknown. Walter [4] recently showed that a key equation in [1] had an error in sign, undetected by Sommerfeld or other commentators, which M. evidently soon saw. This error had led to omitting the factor $\beta = \left(1 - v^2 / c^2\right)^{-1/2}$ in the relativistic velocity $\mathbf{u} = \mathbf{p} / m_0 = \beta \mathbf{v} = \beta \dot{\mathbf{x}}$. With the error corrected, velocity is still constrained to a negative-curvature 3-space, but space-time is a flat 4-space. The changes between [1], [2] and [3] reflect Minkowski's evolving understanding, his different intended audiences, and the likelihood that he chose to defer the noneuclidean aspects of velocity and of space-time for later treatment. His sudden death prevented that. Had the cosmological expansion been known in 1907, his treatment of these symmetries might have been very different.

We have only 3 published documents written by Minkowski as contributions to relativity, all written within 14 months before his sudden death on Jan. 12, 1909. The first [1] is the text of a lecture he gave to mathematicians at Göttingen on Nov. 5, 1907. He never published it, but a few years after he died Sommerfeld arranged for its publication. In it he announced prominently that the world is "in a certain sense a four-dimensional, noneuclidean manifold," and supported that by showing that the geometry of relativistic velocity space is noneuclidean. Only 6 weeks later he submitted for publication a long paper [2], containing most of the results in [1] and much more. It includes prominently the equations showing that velocity space has noneuclidean negative curvature, but does not mention these geometric implications in the verbal text. The third paper [3] is his most famous, a major lecture he presented at a German scientific congress in September, 1908, devoted to the concept of four-dimensional space-time. It does not mention the noneuclidean aspect at all. Historians have long discussed the resulting puzzle: Why did Minkowski never discuss in the papers [2,3] that he published some of the most prominent conclusions of his first lecture, those dealing with noneuclidean geometry? (They all turned out to be true.)

He seems to have had a plausible reason to hesitate. Ten years ago Scott Walter [4] discovered an error in the sign of one term in the equations for the velocity four-vector in [1]. That error does not contaminate what the lecture says about the noneuclidean geometry of velocity space; that is perfectly correct, but an analysis shows that the velocity equations supporting it are incomplete in other ways as well. The velocity paragraph in [1] seems to have been written just at the moment when Minkowski was discovering the true covariant form of the velocity vector, while he did not yet have all the details correct. At the same time, the form of those equations brought him the sudden insight that all this was a brand new case of a mathematically old story, the noneuclidean geometry of Lobachevsky and Bolyai.

In addition to the errors and incompleteness of the new velocity derivation, the text of [1] gives other evidence that the velocity paragraph is a very late and hasty insertion. It breaks into the continuity of a well-designed and systematic brief presentation of much of the material of electrodynamics, presented in a new way relying on Minkowski's four-dimensional symmetry coordinates. It appears that this new insight about the velocity and its geometry—a break-through—had come to Minkowski very close to the date of the lecture, and he thought it so important that he added it to his text even though it was not complete.

On this hypothesis, it is now possible to give a new reading to this important text. The more mature earlier text of the lecture can now be separated from the new and less well ripened inserted material, and the latter can be examined on its own. It sheds much light on Minkowski's creative process—he makes a rough guess at what he thinks the form of the result should be, tests it against other expectations he also has, makes new adjustments, and learns from this whole process as he goes along. In his mathematical work, Minkowski was well known for his powerful geometric insight and imagination; it is now possible to see how he brings this to bear on the velocity problem, and how it helps bring him to his noneuclidean insight.

After the lecture, of course, he continued the process. For the velocity four-vector he must have arrived almost immediately at the true result, which he presented clearly in the equations of [2], but without any textual discussion of the geometrical implications. That did not need to be said. The mathematical education of the time, even for theoretical physicists, included noneuclidean geometry and its algebra. On May 18, 1908, only a few months after [2] was published, Einstein's collaborator Jakob Laub writes to him from Tübingen [5] and comments skeptically on the infatuation of the theoretical physicist M. Cantor with aspects of the Minkowski formalism, saying, "I think he has let himself be impressed by the noneuclidean geometry."

By the time he was writing the paper [2], as Walter has noted, Minkowski was clearly aware of the contrast between the relativistic velocity vector, which was constrained to a negatively curved 3-space, and the position vector, inseparably bound up in a flat 4-space. The metrics for both these spaces share the well-known $(+++-)$ metric signature. It is likely that the new questions raised by this near and yet incomplete symmetry between velocity and space-time led Minkowski to hold back on opening up this additional issue in the papers [2,3], which had so much else that was new to teach. He may very well have expected to return to these questions in the future. He did not change his mind, but death prevented him from explaining what he knew.

References:

[1] Minkowski, H., Ann. Phys. (4) **47**, 927 (1915).
[2] Minkowski, H., Gött. Nach. (1908) 53.
[3] Minkowski, H., Phys. Zeitschr. **10**, 104 (1909).
[4] Walter, S., The noneuclidean style of Minkowskian relativity. In Jeremy Gray, ed., *The Symbolic Universe: Geometry and Physics, 1890-1930*, 91-127 (Oxford, 1999).
[5] Laub, J., in Einstein, A., Collected Papers, Vol. 5, p. 119.